Comment on "Maximum of the Local Entropy Production Becomes Minimal in Stationary Processes"

Struchtrup and Weiss [1] (SW) proposed a new variational principle for stationary nonequilibrium flows, such as the steady boundary-driven flows of mass and heat shown in Fig. 1. SW suggest that of all possible flows satisfying the specified boundary conditions, the solution with the smallest maximum local entropy production is correct. They suggest that the minimax principle could be extended to nonstationary flows. SW consider the one-dimensional heat flow of a Boltzmann gas. They calculate the local entropy production and express its maximum as a function of the dimensionless heat flux $\hat{q}$. They show that the heat flux corresponding to the minimum of the maximum local entropy production density also corresponds to the solution of this linear transport problem.

Despite their success with this simple example problem, we believe that more complex, nonlinear problems are required to test any assertion of generality for such a principle. Three potential limitations for the validity of the principle are the following: (i) The principle ignores the possible coexistence of stable multiple solutions for the same boundary conditions; (ii) the principle fails to treat the overall stability of two disparate, but weakly coupled, nonequilibrium systems; (iii) the principle excludes transitions to more stable states, such as the transition from quiescent Fourier conduction to the steady convective state which produces entropy at a greater rate [2,3].

To make our reservations more explicit, we display a set of stationary, convecting flows for a compressible, heat-conducting viscous fluid described by the Navier-Stokes equations. The flows have identical boundary conditions with a Rayleigh number of 40 000. The sides of the simulated cell are periodic and the width is twice the height. The 2- and 4-roll flows are stable attractors in the solution space while the more complex 6-roll solution is unstable. For details, see Ref. [4].

The local internal entropy production [5,6] for the compressible Navier-Stokes equations is

$$g_i \equiv (\overline{\overline{\sigma}}: \nabla \vec{u} - \vec{q} \cdot \nabla \ln T)/T ,$$

where $\overline{\overline{\sigma}}$ is the stress tensor and $\vec{q}$ is the heat flux. This expression for the local entropy production is validated by confirming that the integral $\dot{S}_i = \int_V g_i \, dV$ is precisely equal to the total flow of entropy at the boundaries, $\dot{S}_i = \dot{S}_e \equiv \frac{Q}{T_H} - \frac{Q}{T_C}$. T_H and T_C are the temperatures of the hot and cold boundaries, and $Q \equiv \int_A \vec{q} \cdot dA$.

The maxima in the local entropy productions, per unit volume, ($\max[g_i]$) for the (2,4,6)-roll flows are, respectively, 0.007 03, 0.006 24, and 0.005 55. Thus two or more stable solutions (2- and 4-roll) exist, for the same boundary conditions but with different $\max[g_i]$. It is amusing that an unstable "solution," with six rolls, has the smallest $\max[g_i]$ of the three cases considered here.

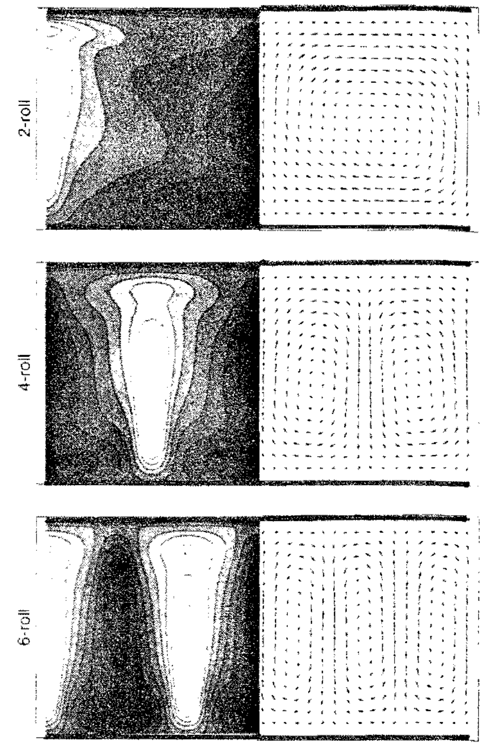

FIG. 1. For the (2,4,6)-roll flows, the local entropy production (left) and velocity field (right) is shown. The region of maximum local entropy production (white) corresponds to compression of the cooler fluid.

This work was performed under the auspices of the United States Department of Energy through University of California Contract No. W-7405-Eng-48.

V. M. Castillo and Wm. G. Hoover
LLNL and UC Davis
Livermore, California 94551-7808

Received 2 July 1998 [S0031-9007(98)07915-4]
PACS numbers: 47.15.Fe, 05.70.Ln, 47.20.Bp, 65.50.+m

[1] H. Struchtrup and W. Weiss, Phys. Rev. Lett. **80**, 504 (1998).
[2] Lord Rayleigh, Philos. Mag. **32**, 529 (1916).
[3] S. Chandrasekhar, *Hydrodynamic and Hydromagnetic Stability* (Oxford, London, 1961).
[4] V. M. Castillo, Wm. G. Hoover, and C. G. Hoover, Phys. Rev. E **55**, 5546 (1997).
[5] R. B. Bird, W. E. Stewart, and E. N. Lightfoot, *Transport Phenomena* (Wiley, New York, 1960).
[6] V. M. Castillo and Wm. G. Hoover, Phys. Rev. E **58**, 735 (1998).

 0031-9007/98/81(25)/5700(1)$15.00

) and Weiss Reply: In our Letter [1] we have a minimax principle which we used for the on of boundary conditions for moment systems l th dynamics. This principle is criticized and Hoover [2] on the grounds of solutions er-Stokes equations. These authors consider a problem which has several solutions, and one unstable. It appears that the unstable solution llest maximum of the local entropy production olutions that are obtained. Thus, the minimax hooses the unstable solution and Castillo and ;gest there that it may not be true. However, *need for the minimax principle in the Navier-ry*, since all boundary conditions are known. so in contrast to extended thermodynamics. a system of equations of many moments atics requires more boundary conditions than ı are controlled—and controllable—in an ex-We have used the principle for the determina-litional boundary conditions which cannot be *priori.*

-dimensional stationary heat transfer in the e which is considered in our Letter [1] is st case where this problem occurs: The s and velocities at two walls and the pressure lled in an appropriate experiment, but the s of the problem requires the prescription litional boundary condition. This additional foll from the minimax principle which for tion ay be formulated as follows: nary processes boundary values which are led in the experiment assume those values antee that the maximum of the local entropy becomes minimal.

lication of the minimax principle is not recog-e Comment of Castillo and Hoover. The au-ever, consider the compressible Navier-Stokes which require only controllable boundary con-the one-dimensional case, for instance, the es equations require four boundary conditions. iment one may control two temperatures and ies—no additional boundary conditions are or the Navier-Stokes case and there is no need imax principle. In fact, there is no possibility ication, since the boundary conditions do not nal degrees of freedom. In their problem they solutions and ask whether the minimax prin-e used as an indicator for the stability of the They show that the maximum entropy produc-decreases with the complexity of the convec-. and that its minimum characterizes an unstable

and Hoover emphasize that the maximum duc n of the unstable solution is smaller of stable solutions. In this context it should be noted that unstable solutions with low entropy production are realizable in experiments. In Poiseuille flow one usually finds laminar flow with low entropy production for Reynolds numbers below 2300 only. In extremely undisturbed flows, however, it is possible to realize laminar flow with Reynolds numbers up to 50 000.

Of course, if there exist several solutions for a given set of boundary conditions, one has to perform a stability analysis in order to find stable solutions. We agree with Castillo and Hoover that the minimax principle cannot replace the stability analysis, and we like to emphasize that we did not suggest its application in the context of stability analysis.

It should be mentioned that in moment systems a stabil-ity analysis may be required together with the application of the minimax principle. This will be the case when the maximum of the local entropy production as a function of the additional boundary conditions (which for the 14-field case was called $\hat{\Sigma}_{\max}(\hat{q})$ in [1]) has several minima or if the function is not unique. Until now, we did not study such a case, but it should be expected to occur in convection flows described by extended moment methods.

Let us summarize our discussion in three statements as follows:

(1) The minimax principle cannot be used for the determinations of boundary conditions for the Navier-Stokes equations, since all necessary boundary conditions are controlled in experiment.

(2) The minimax principle does not replace any stability criterion and should not be used for stability analysis.

(3) The future description of convection flows by extended moment methods will require the use of the minimax principle and a stability analysis.

Castillo and Hoover write in their Comment [2] that we "suggested that (the) minimax principle could be extended to nonstationary flows." We take the opportunity of this Reply to state that the principle is restricted to stationary processes only. In [1] this was not expressed as clearly as it should have been.

Henning Struchtrup
Dipartimento di Matematica, Università di Catania
Viale Andrea Doria 6, 95125 Catania, Italy

Wolf Weiss
Technical University Berlin, FB 6, Sekretariat HF2
Strasse des 17. Juni 135, D-10623 Berlin, Germany

Received 11 August 1998 [S0031-9007(98)07914-9]
PACS numbers: 47.27.Cn, 05.70.Ln, 47.15.Fe, 47.27.Eq

[1] H. Struchtrup and W. Weiss, Phys. Rev. Lett. **80**, 5048 (1998).
[2] V. M. Castillo and Wm. G. Hoover, preceding Comment, Phys. Rev. Lett. **81**, 5700 (1998).

0031-9007/98/81(25)/5701(1)$15.00